J. Fluid Mech. (1995), vol. 304, pp. 263–283

On hydromagnetic instabilities driven by the Hartmann boundary layer in a rapidly rotating sphere

By KEKE ZHANG[1] AND F. H. BUSSE[2]

[1]Department of Mathematics, University of Exeter, UK

[2]Institute of Physics, University of Bayreuth, Germany

(Received 14 July 1994 and in revised form 19 July 1995)

The instability of an electrically conducting fluid of magnetic diffusivity λ and viscosity ν in a rapidly rotating spherical container of magnetic diffusivity $\hat{\lambda}$ in the presence of a toroidal magnetic field is investigated. Attention is focused on the case of a toroidal magnetic field induced by a uniform current density parallel to the axis of rotation, which was first studied by Malkus (1967). We show that the internal ohmic dissipation does not affect the stability of the hydromagnetic solutions obtained by Malkus (1967) in the limit of small λ. It is solely the effect of the magnetic Hartmann boundary layer that causes instabilities of the otherwise stable solutions. When the container is a perfect conductor, $\hat{\lambda} = 0$, the hydromagnetic instabilities grow at a rate proportional to the magnetic Ekman number of the fluid E_λ; when the container is a nearly perfect insulator, $\lambda/\hat{\lambda} \ll 1$, the hydromagnetic instabilities grow at a rate proportional to $E_\lambda^{1/2}$; when the container is a nearly perfect conductor, $\hat{\lambda}/\lambda \ll 1$, the growth rates are proportional to $\hat{E}_\lambda{}^{1/2}$, where $\hat{E}_\lambda$ is the magnetic Ekman number based on the diffusivity $\hat{\lambda}$ of the container. The main characteristics of the instabilities are not affected by varying magnetic properties of the container. In light of the destabilizing role played by the Hartmann boundary layer, we also examine the corresponding magnetoconvection in a rapidly rotating fluid sphere with the perfectly conducting container and stress-free velocity boundary conditions. Analytical magnetoconvection solutions in closed form are obtained and implications are discussed.

1. Introduction

It is well-known that the geomagnetic field undergoes variations on the characteristic time scales of decades to centuries (the geomagnetic secular variation) (Jacobs 1975). A particular feature of the geomagnetic secular variation that has received much attention is the west drift of about 0.2° of longitude per year of the non-axisymmetric component of the field. An important attempt was first made by Hide (1966) to interpret this feature as a manifestation of hydromagnetic waves in association with the toroidal magnetic field in the Earth's fluid core (see also Hide & Stewartson 1972). Assuming a plausible strength of a uniform toroidal field, making use of the Rossby β-plane approximation and neglecting ohmic dissipation in a thin rotating spherical shell, Hide considered hydromagnetic waves as perturbations of the basic toroidal field. Malkus (1967) proposed and studied a non-uniform toroidal magnetic field induced by a uniform current density along the axis of rotation parallel to the unit vector $\hat{\boldsymbol{k}}$

$$\boldsymbol{B}_0 = (B_0/r_0)\hat{\boldsymbol{k}} \times \boldsymbol{r}, \tag{1.1}$$

where B_0 is the maximum strength of the field in the sphere, $\boldsymbol{r}$ is a position vector and r_0 is the radius of the fluid sphere. Malkus considered slight perturbations to the motionless state with the magnetic field (1.1) in a perfectly conducting, inviscid rotating fluid sphere and found that solutions describing hydromagnetic waves can be obtained. Quite remarkably, Malkus was able to reduce the analysis of the problem to that of the corresponding non-magnetic problem. Since then the magnetic field (1.1) has been extensively studied with or without diffusive effects in different geometries owing largely to the fact that the Malkus' field model leads to a considerable mathematical simplification. With a local analysis, Acheson (1972, 1978) demonstrated that the field (1.1) cannot be field-gradient unstable. In order that a toroidal magnetic field $B(s)$ becomes unstable through the mechanism of the field-gradient instability, where s is the distance from the axis of rotation, $B(s)$ has to increase somewhere faster than $s^{3/2}$,

$$\varDelta = \frac{s}{(B(s)/s)^2}\frac{\mathrm{d}}{\mathrm{d}s}(B(s)/s)^2. \tag{1.2}$$

For the magnetic field (1.1), $\varDelta = 0$. The results of Acheson were confirmed by extensive numerical analyses in cylindrical geometries by Fearn (for examples, see Fearn 1983).

It was first noticed by Roberts & Loper (1979) that the hydromagnetic waves with the azimuthal wavenumber $m = 1$ produced by the field (1.1) in an axisymmetric container with $\hat{\lambda} = 0$, where $\hat{\lambda}$ is the magnetic diffusivity of the container, can be diffusively unstable. Without using explicit solutions of the zero-order problem, they were able to derive an instability criterion: the field (1.1) may be unstable to westwardly propagating hydromagnetic waves for any non-zero magnetic field strength if

$$0.5 < \frac{I^2 m - \omega_0}{I^2 m^2 - \omega_0^2} < 1, \tag{1.3}$$

where $I^2 \sim B_0^2$, m is the azimuthal wavenumber and ω_0 is the frequency of the waves (see §2 for details). But it should be noted that the instability criterion (1.3) can be only applied to the case $\hat{\lambda} = 0$ in which magnetic boundary-layer solutions are not needed. As long as $\hat{\lambda} \neq 0$, both the explicit solution of the zero-order problem and the complete boundary-layer solution are required in order to determine the stability of the field (1.1). Roberts & Loper (1979) found that the solutions of hydromagnetic waves that satisfy the criterion (1.3) are stable within a sphere when $\hat{\lambda} \neq 0$ including $\hat{\lambda} \to 0$. No unstable modes in a spherical container with $\hat{\lambda} \neq 0$ were found in their analysis. Roberts & Loper (1979) attributed the stability of the field (1.1) in a sphere to the stabilizing influence of the spherical boundaries which, unlike for a cylinder, are not parallel to the axis of rotation. Furthermore, it was not recognized that the magnetic spherical boundary layer can be the sole cause of magnetic instabilities.

A detailed investigation of the magnetic field (1.1) in a rotating fluid layer, including the effects of both magnetic and thermal diffusion, was carried out by Soward (1979). The analysis of thermal instabilities with the presence of the field (1.1) was then extended to a rapidly rotating fluid sphere containing a uniform distribution of heat sources (Eltayeb & Kumar 1977; Fearn 1979*a,b*; Eltayeb 1992). Kerswell (1994) has recently used the field (1.1) to examine the tidal excitation of hydromagnetic waves in a rotating spheroid like the realistic fluid core of the Earth. It is perhaps not an overstatement that the Malkus field model (1.1) is the most extensively studied model of hydromagnetic processes in rotating systems. It should be mentioned, though, that many important instabilities are filtered out by the magnetic field (1.1) and other more realistic magnetic fields in spherical geometry have therefore been examined (Fearn & Proctor 1983; Fearn & Weiglhofer 1991; Zhang & Fearn 1993, 1994). A more complete discussion and additional references may be found in two recent review papers by Fearn (1993) and Proctor (1994).

A point of central importance to the present analysis is that the hydromagnetic solutions obtained from a perfectly conducting fluid (for example, Malkus 1967) cannot satisfy all magnetic boundary conditions that are required at the interface between the fluid and the container. A Hartmann boundary layer is thus needed in order that all the necessary magnetic boundary conditions are satisfied. The vital importance of Hartmann layers to the problem of hydromagnetic instability has not been much discussed. The primary aim of this paper is to examine the effects of the spherical Hartmann boundary layer on the instabilities of the dynamically stable hydromagnetic wave solutions studied by Malkus (1967, 1968). Our discussion mainly focuses on the following five cases. First, we extend the analysis of Roberts & Loper (1979) (viscosity $\nu = 0$) by including the effects of the Ekman boundary layer ($\nu \neq 0$). It is stressed that, even though the explicit boundary solutions are not needed when perfectly conducting and stress-free boundary conditions are assumed, it is still solely the magnetic Hartmann boundary layer that causes instability of the field (1.1). We then examine similar instabilities in a nearly perfectly conducting spherical container, a nearly insulating container and a container of arbitrary conductivity. Finally, we investigate the problem of the corresponding magnetoconvection in a rapidly rotating sphere by assuming that the container is a perfect conductor and that velocity boundary conditions are stress-free. Complete analytical magnetoconvection solutions in closed form are then obtained in the limit of small Prandtl number. The main conclusion of the paper is that, driven solely by the magnetic Hartmann boundary layer, the Malkus magnetic field (1.1) is always unstable in a rapidly rotating sphere regardless of the magnetic boundary condition in the limit of vanishing viscosity of the fluid.

The remainder of the paper is organized as follows. Perturbation analysis and the formulation of boundary-layer solutions are presented in §2. This is followed by a discussion of hydromagnetic instability for different conductivities of the container. In §4, we examine the problem of magnetoconvection in a rapidly rotating sphere. The paper closes with a summary of the main results and some remarks in §5.

2. Perturbation analysis and the Hartmann layer

2.1. *Governing equations*

Consider an electrically conducting fluid of constant magnetic diffusivity λ, kinematic viscosity ν and density ρ. The fluid is enclosed in a spherical container of inner radius r_o with magnetic diffusivity $\hat{\lambda}$ and the whole system rotates rapidly with a constant angular velocity $\Omega\hat{\boldsymbol{k}}$. To write the dimensionless equations, we let

$$\to r_o r,\ \ t \to t\Omega^{-1},\ \ \boldsymbol{B} \to (\rho\mu)^{1/2}\Omega r_0 \boldsymbol{B},\ \ P \to P\rho r_0 \Omega^2,$$

for length, time, magnetic field and pressure respectively. The fluid motions are then governed by the following dimensionless equations:

$$\left(\frac{\partial}{\partial t} - E_\nu \nabla^2\right) \boldsymbol{u} + 2\hat{\boldsymbol{k}} \times \boldsymbol{u} = -\nabla P + (\nabla \times \boldsymbol{B}) \times \boldsymbol{B}, \tag{2.1}$$

$$\nabla \cdot \boldsymbol{u} = 0, \quad \nabla \cdot \boldsymbol{B} = 0, \tag{2.2a, b}$$

$$\left(\frac{\partial}{\partial t} - E_\lambda \nabla^2\right) \boldsymbol{B} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}), \tag{2.3}$$

while the magnetic field in the container, $\hat{\boldsymbol{B}}$, is described by

$$\left(\frac{\partial}{\partial t} - \hat{E}_\lambda \nabla^2\right) \hat{\boldsymbol{B}} = 0. \tag{2.4}$$

The non-dimensional parameters – the magnetic Ekman number E_λ for the fluid, the magnetic Ekman number $\hat{E}_\lambda$ for the container and the Ekman number E_ν – are defined by

$$E_\lambda = \frac{\lambda}{\Omega r_o^2}, \quad \hat{E}_\lambda = \frac{\hat{\lambda}}{\Omega r_o^2}, \quad E_\nu = \frac{\nu}{\Omega r_o^2}.$$

In view of the application to the Earth's liquid core, viscous dissipation is likely to be much smaller than magnetic dissipation,

$$\epsilon = \frac{E_\nu}{E_\lambda} \ll 1.$$

Consequently, we shall in some cases neglect the term $E_\nu \nabla^2 \boldsymbol{u}$ in equation (2.1). It follows that the velocity boundary condition simply requires

$$\boldsymbol{u} \cdot \hat{\boldsymbol{r}} = 0 \tag{2.5}$$

at $r = 1$, where (r, θ, ϕ) are spherical coordinates with unit vectors $(\hat{\boldsymbol{r}}, \hat{\theta}, \hat{\phi})$. We shall also assume that both magnetic Ekman numbers are small: $\hat{E}_\lambda \ll 1$ and $E_\lambda \ll 1$. A boundary-type solution is thus required not only for the magnetic field $\boldsymbol{B}$ in the fluid adjacent to the container but also for the magnetic field $\hat{\boldsymbol{B}}$ in the container close to the interface between the fluid and container. The boundary conditions at the interface are that the magnetic field and the tangential component of the electrical field are continuous:

$$\boldsymbol{B} = \hat{\boldsymbol{B}}, \quad \hat{\boldsymbol{r}} \times \boldsymbol{E} = \hat{\boldsymbol{r}} \times \hat{\boldsymbol{E}}, \quad r = 1. \tag{2.6a, b}$$

2.2. *Perturbation analysis*

This paper concentrates on the linear stability of the Malkus field (1.1) which may be written in the dimensionless form

$$\boldsymbol{B}_0 = I \hat{\boldsymbol{k}} \times \boldsymbol{r} \tag{2.7}$$

with

$$I = \frac{\Omega_A}{\Omega},$$

where Ω_A is the Alfvén angular frequency. Since it is well-known that axisymmetric perturbation is of secondary importance to the problem of hydromagnetic instability, we look for solutions in the form of the azimuthally travelling waves

$$[\boldsymbol{u}, (\boldsymbol{B} - \boldsymbol{B}_0), \hat{\boldsymbol{B}}, P] = [\boldsymbol{u}, (\boldsymbol{B} - \boldsymbol{B}_0), \hat{\boldsymbol{B}}, P](\theta, r) \mathrm{e}^{\mathrm{i}(m\phi + \omega t)}$$

of the linearized equations (2.1–2.3) with azimuthal wavenumbers $m \geqslant 1$. We assume that solutions of these equations for an arbitrarily small but non-zero E_λ can be written as

$$\boldsymbol{u} = \boldsymbol{u}_0 + (\boldsymbol{u}_b + \boldsymbol{u}_i), \tag{2.8a}$$

$$P = P_0 + (P_b + P_i), \tag{2.8b}$$

$$\boldsymbol{B} - \boldsymbol{B}_0 = \boldsymbol{b}_0 + (\boldsymbol{b} + \boldsymbol{b}_i), \tag{2.8c}$$

$$\omega = \omega_0 + \omega_1, \tag{2.8d}$$

where $(\boldsymbol{u}_0, \boldsymbol{b}_0, P_0, \omega_0)$ represent the solution for a perfectly conducting fluid with $E_\lambda = 0$. The perturbations introduced by an arbitrarily small, but nonzero E_λ are divided into the small interior perturbations, $\boldsymbol{u}_i, \boldsymbol{b}_i$ and P_i:

$$\boldsymbol{u}_i = O(|\boldsymbol{u}_0| E_\lambda^{1/2}), \quad \boldsymbol{b}_i = O(|\boldsymbol{b}_0| E_\lambda^{1/2}), \qquad P_i = O(P_0 E_\lambda^{1/2}),$$

and the boundary-layer corrections, $\boldsymbol{u}_b, \boldsymbol{b}$ and P_b, which are non-zero only in the Hartmann boundary layer. A small perturbation to ω_0 is denoted by ω_1. After substituting expansions (2.8a–d) into equations (2.1–2.3) and linearizing the resulting equations, we obtain at leading order ($E_\lambda = 0$)

$$\mathscr{L}[\boldsymbol{u}_0, \pi_0] = \mathrm{i}\omega_0 \left(1 - \frac{m^2 I^2}{\omega_0^2}\right) \boldsymbol{u}_0 + 2\left(1 - \frac{mI^2}{\omega_0}\right) \hat{\boldsymbol{k}} \times \boldsymbol{u}_0 + \nabla \pi_0 = 0, \tag{2.9}$$

$$\nabla \cdot \boldsymbol{u}_0 = 0, \quad \nabla \cdot \boldsymbol{b}_0 = 0, \tag{2.10a,b}$$

$$\boldsymbol{b}_0 = \frac{mI}{\omega_0} \boldsymbol{u}_0, \tag{2.11}$$

together with boundary conditions $\boldsymbol{u}_0 \cdot \hat{\boldsymbol{r}} = 0$ and $\boldsymbol{b}_0 \cdot \hat{\boldsymbol{r}} = 0$ at $r = 1$, where

$$\pi_0 = P_0 + I(\hat{\boldsymbol{k}} \times \boldsymbol{r}) \cdot (I\hat{\boldsymbol{k}} \times \boldsymbol{r} + \boldsymbol{b}_0).$$

There exist apparently ∞^3 possible solutions, $(\boldsymbol{u}_0, \boldsymbol{b}_0, P_0, \omega_0)$, for equations (2.9–2.11). In order to study the instability characteristics of the field (2.7), it is necessary to select some appropriate classes of solutions among the entire manifold of solutions, which can provide a sufficient condition for instability. Malkus (1967) showed that equations (2.9–2.11) lead to a modified Poincaré equation. A detailed discussion of the Poincaré equation may be found in the books by Greenspan (1969) and Lyttleton (1953). Solutions of equations (2.9–2.11) are, in general, characterized by three different indexes, (l, m, n): m is the azimuthal wavenumber, n indicates the structure in the s-direction and the integer l, $l > m$, usually represents the degree of complexity of solutions in the z-direction, where cylindrical coordinates (s, ϕ, z) are used. Following Malkus (1967, 1968), we select the two classes of solutions of equations (2.9–2.11) that correspond to the simplest structure of the fluid motions along the direction of the axis of rotation. This selection is based on an expectation that the modes with complicated z-structure in realistic rapidly rotating systems, without or with the influence of magnetic fields, are of secondary significance (Busse 1970, 1982; Zhang 1995b). The first class is referred to as equatorially anti-symmetric waves (modes) ($l - m = 1$, for details see Zhang 1993) with the following equatorial symmetry:

$$(u_s, u_z, u_\phi)(s, z, \phi) = (-u_s, u_z, -u_\phi)(s, -z, \phi),$$

and the second class will be called equatorially symmetric waves (modes) ($l-m=2$) with

$$(u_s, u_z, u_\phi)(s,z,\phi) = (u_s, -u_z, u_\phi)(s,-z,\phi).$$

There are no restrictions on the value of azimuthal wavenumber m and the maximum value of index n is determined by the eigenvalue relation (Greenspan 1969). It is also worth noting that the second class of solutions represents, at leading order, the convection solutions for fluids with small Prandtl number (Zhang 1994).

Malkus (1967) also showed that solutions of equations (2.9)–(2.11) in the form of azimuthally travelling waves may become dynamically unstable if I is sufficiently large. But the critical value, $I_0 = O(1)$, required for this type of instability is too large to be geophysically relevant (Roberts & Loper 1979; Fearn 1993; Proctor 1994). This paper is only concerned with hydromagnetic instabilities in the interval $I < I_0$, where the magnetic field (2.7) is stable without the effects of ohmic dissipation.

In studying the diffusive instability of the dynamically stable hydromagnetic modes, we must carry the perturbation problem to the next order by adding ohmic dissipation to the system, including both internal and boundary dissipation. Governing equations at the next-order perturbation can be readily obtained by substituting expansions (2.8a–d) into equations (2.1–2.3) and subtracting equations (2.9–2.11)

$$\mathrm{i}\omega_0 \boldsymbol{u}_1 + 2\hat{\boldsymbol{k}} \times \boldsymbol{u}_1 + \nabla P_1 = 2I\hat{\boldsymbol{k}} \times \boldsymbol{b}_1 + \mathrm{i}mI\boldsymbol{b}_1 - \mathrm{i}\omega_1 \boldsymbol{u}_0, + E_v \nabla^2(\boldsymbol{u}_0 + \boldsymbol{u}_1), \tag{2.12}$$

$$\nabla \cdot \boldsymbol{u}_1 = 0, \quad \nabla \cdot \boldsymbol{b}_1 = 0, \tag{2.13a,b}$$

$$(\mathrm{i}\omega_0 - E_\lambda \nabla^2)\boldsymbol{b}_1 - \mathrm{i}mI\boldsymbol{u}_1 = E_\lambda \nabla^2 \boldsymbol{b}_0 - \mathrm{i}\omega_1 \boldsymbol{b}_0, \tag{2.14}$$

where $\boldsymbol{b}_1 = \boldsymbol{b}_i + \boldsymbol{b}$, $P_1 = P_i + P_b$ and $\boldsymbol{u}_1 = \boldsymbol{u}_i + \boldsymbol{u}_b$. After some manipulation, equations (2.12–2.14) may be combined into a single differential equation

$$\mathscr{L}[\boldsymbol{u}_1, \pi_1] = \frac{-\mathrm{i}\omega_1 \boldsymbol{u}_0}{\omega_0}\left[\omega_0 + \frac{I^2 m^2}{\omega_0} - \frac{I^2 m(\omega_0^2 - I^2 m^2)}{\omega_0(\omega_0 - I^2 m)}\right]$$

$$+\frac{IE_\lambda}{\omega_0}[-2\mathrm{i}\hat{\boldsymbol{k}} \times \nabla^2(\boldsymbol{b}_0 + \boldsymbol{b}) + m\nabla^2(\boldsymbol{b}_0 + \boldsymbol{b})] + E_v \nabla^2(\boldsymbol{u}_0 + \boldsymbol{u}_b), \tag{2.15}$$

where

$$\pi_1 = \frac{I^2 m \omega_1 \pi_0}{\omega_0(\omega_0 - I^2 m)} + P_1,$$

and where the small terms $\nabla^2 \boldsymbol{u}_i$ and $\nabla^2 \boldsymbol{b}_i$ in equation (2.15) have been neglected. It should be noted that, in writing equations (2.12) and (2.14), we have assumed that $I^2 \gg E_\lambda^{1/2}$, or more precisely, $I = O(E_\lambda^{\alpha/4})$, $0 \leqslant \alpha < 1$. Note also that the magnetic field $(\boldsymbol{b}_0 + \boldsymbol{b})$ needs to be matched to solutions of equation (2.4) at the interface between the fluid and the container, which is to be discussed in the next section.

The inhomogeneous differential equation (2.15) must satisfy a certain solvability condition in order that it is solvable. We may obtain the solvability condition by multiplying equation (2.15) by the complex conjugate of $\boldsymbol{u}_0$, denoted by $\boldsymbol{u}_0^*$, and integrating over the volume of the sphere. Using the fact

$$\int_V \boldsymbol{u}_0^* \cdot \mathscr{L}[\boldsymbol{u}_1, \pi_1]\mathrm{d}V = \int_S \hat{\boldsymbol{r}} \cdot (\pi_1 \boldsymbol{u}_0^* + P_0^* \boldsymbol{u}_1)\mathrm{d}S = 0, \tag{2.16}$$

where $\int_V$ denotes the volume integral over the sphere and $\int_S$ represents the surface integral over the spherical surface at $r = 1$, we obtain

$$\mathrm{i}\omega_1\left[-1-\frac{I^2m^2}{\omega_0^2}+\frac{I^2m(\omega_0^2-I^2m^2)}{\omega_0^2(\omega_0-I^2m)}\right]\int_V|\boldsymbol{u}_0|^2\mathrm{d}V=-\frac{ImE_\lambda}{\omega_0}\mathscr{I}_1+\frac{2IE_\lambda}{\omega_0}\mathscr{I}_2-E_v\mathscr{I}_3, \tag{2.17}$$

where

$$\mathscr{I}_1=\int_V\boldsymbol{u}_0^*\cdot\nabla^2(\boldsymbol{b}_0+\boldsymbol{b})\mathrm{d}V,\quad \mathscr{I}_2=\mathrm{i}\int_V\boldsymbol{u}_0^*\cdot\hat{\boldsymbol{k}}\times\nabla^2(\boldsymbol{b}_0+\boldsymbol{b})\mathrm{d}V,\quad \mathscr{I}_3=\int_V\boldsymbol{u}_0^*\cdot\nabla^2\boldsymbol{u}_b\mathrm{d}V.$$

Since $\mathrm{i}\omega_1 = \mathrm{i}\mathrm{Re}[\omega_1] + \sigma$, σ being the growth rate of disturbances, equation (2.17) leads to

$$\sigma=\frac{E_\lambda(\omega_0-I^2m)}{\omega_0(\omega_0^2-2I^2m\omega_0+I^2m^2)}\mathrm{Re}[I\omega_0m\mathscr{I}_1-2I\omega_0\mathscr{I}_2+\epsilon\omega_0^2\mathscr{I}_3]\Big/\int_V|\boldsymbol{u}_0|^2\mathrm{d}V \tag{2.18}$$

for a given wavenumber m and a given value of I. With the selected solutions $\boldsymbol{u}_0$, this equation provides a sufficient criterion for instability: the field (2.7) is unstable if the resulting growth rate σ is positive.

An important result which can be immediately shown is that

$$\sigma \equiv 0$$

for any values of m, ω_0 and I ($I < I_0$) if the effects of the boundary-layer dissipation are neglected through the assumption $\boldsymbol{b} = 0$ and $\boldsymbol{u}_b = 0$ in (2.18). This result follows from the fact that the velocity $\boldsymbol{u}_0^*$ is orthogonal to $\nabla^2\boldsymbol{b}_0$ and $\nabla^2\boldsymbol{u}_0$:

$$\int_V\boldsymbol{u}_0^*\cdot\nabla^2\boldsymbol{b}_0\mathrm{d}V=\frac{4mI(\omega_0-Im^2)^2}{\omega_0(\omega_0^2-I^2m^2)^2}\int_V\boldsymbol{u}_0^*\cdot(\hat{\boldsymbol{k}}\cdot\nabla)^2\boldsymbol{u}_0\mathrm{d}V=0$$

(see Zhang 1994 for details) and that the second integral $\mathscr{I}_2$ can be reduced to

$$\mathrm{i}\int_V\boldsymbol{u}_0^*\cdot(\hat{\boldsymbol{k}}\times\nabla^2\boldsymbol{b}_0)\mathrm{d}V=\frac{4mI(\omega_0-Im^2)^2}{\omega_0(\omega_0^2-I^2m^2)^2}\int_V\mathrm{i}\hat{\boldsymbol{k}}\cdot[(\hat{\boldsymbol{k}}\cdot\nabla)^2\boldsymbol{u}_0\times\boldsymbol{u}_0^*]\mathrm{d}V=0$$

for both the anti-symmetric and symmetric modes. In other words, the stability characteristics are not affected at leading order by internal viscous and ohmic dissipations. Any hydromagnetic instabilities for $I < I_0$, if they exist, must be driven by the diffusive processes taking place in the thin hydromagnetic boundary layer. There are three different diffusive processes represented by the three integrals $\mathscr{I}_j, j = 1, 2, 3$, through which we may identify the mechanism of hydromagnetic instability. First we note in equation (2.18) that

$$\frac{\omega_0-I^2m}{\omega_0(\omega_0^2-2I^2m\omega_0+2I^2m^2)}>0$$

regardless of the sign of ω_0 for a sufficiently small value of I^2 and $\omega_0 = O(1)$. Note also that we are only interested in small amplitudes of the field (2.7). The existence of an instability is then determined by the sign and relative size of the three boundary integrals. The third integral is related to the Ekman-boundary-layer dissipation resulting from the combined effects of rotation and small viscosity,

$$\mathscr{I}_3=\int_V\boldsymbol{u}_0^*\nabla^2\boldsymbol{u}_b\mathrm{d}V<0,$$

which always provides a sink for the energy and thus a stabilizing influence on the system. The first integral is associated with the magnetic Hartmann boundary layer, a magnetic boundary layer that can exist without rotation,

$$\omega_0 \mathscr{I}_1 = \omega_0 \int_V \boldsymbol{u}_0^* \cdot \nabla^2 \boldsymbol{b} \mathrm{d}V < 0,$$

which also always stabilizes the system. The second integral is also connected with the magnetic Hartmann boundary layer that is strongly influenced by rotation. The sign and size of the second integral

$$-\omega_0 \mathscr{I}_2 = -\omega_0 \int_V \mathrm{i}\boldsymbol{u}_0^* \cdot (\hat{\boldsymbol{k}} \times \nabla^2 \boldsymbol{b}) \mathrm{d}V$$

depend upon both the spatial structure of the boundary layer and the direction of propagation of the corresponding hydromagnetic wave. With a suitable structure and the right direction, the effects of the magnetic Hartmann boundary layer strongly affected by rotation can be destablizing and thus cause instability to the whole otherwise stable hydromagentic system. While detailed boundary-layer solutions are not needed when $\hat{\lambda} = 0$, to make further progress in the case $\hat{\lambda} \neq 0$ the problem of the Hartmann boundary layer must be solved to obtain explicit solutions for $\boldsymbol{b}$. Clearly, the structure of the boundary layer is dependent on the magnetic properties of the container.

2.3. *The Hartmann boundary layer*

Noting that radial derivatives are dominant in the boundary type of solutions, we can derive the leading-order equations governing the Hartmann boundary flow simply by taking dominant terms in equations (2.12–2.14) and setting the interior perturbations $\boldsymbol{b}_i = 0$, $\boldsymbol{u}_i = 0$ and $P_i = 0$,

$$\mathrm{i}\omega_0 \boldsymbol{u}_b + 2\boldsymbol{k} \times \boldsymbol{u}_b + \hat{\boldsymbol{r}}(\hat{\boldsymbol{r}} \cdot \nabla P_b) = 2I\boldsymbol{k} \times \boldsymbol{b} + \mathrm{i}mI\boldsymbol{b}, \tag{2.19}$$

$$\left(\mathrm{i}\omega_0 - E_\lambda \frac{\partial^2}{\partial r^2}\right) \boldsymbol{b} = \mathrm{i}mI\boldsymbol{u}_b, \tag{2.20}$$

where we have assumed that, without loosing the key physics of the problem, $\epsilon \ll 1$, so that the term $E_\nu(\partial^2 \boldsymbol{u}/\partial r^2)$ (the boundary viscous dissipation) is neglected. Solutions of the Hartmann-layer equations (2.19–2.20) must be matched to the magnetic boundary layer solutions outside the fluid sphere governed by

$$\left(\mathrm{i}\omega_0 - \hat{E}_\lambda \frac{\partial^2}{\partial r^2}\right) \hat{\boldsymbol{B}} = 0. \tag{2.21}$$

It is convenient to introduce boundary-layer variables

$$\xi = E_\lambda^{-1/2}(1-r), \quad \hat{\xi} = \hat{E}_\lambda^{-1/2}(r-1),$$

for the Hartmann layer and the exterior magnetic layer respectively. To solve the boundary equations (2.19–2.20), first note that two equations for the magnetic field $\boldsymbol{b}$ can be deduced:

$$\left(\frac{\partial^2}{\partial \xi^2} - \mathrm{i}\omega^*\right) \hat{\boldsymbol{r}} \times \boldsymbol{b} = A\boldsymbol{b}, \tag{2.22}$$

$$\left(\frac{\partial^2}{\partial \xi^2} - \mathrm{i}\omega^*\right) \boldsymbol{b} = -A(\hat{\boldsymbol{r}} \times \boldsymbol{b}), \tag{2.23}$$

where

$$\omega^{\bullet} = \omega_0 + \frac{ml^2(4\cos^2\theta - m\omega_0)}{\omega_0^2 - 4\cos^2\theta}, \quad A = \frac{2ml^2\cos\theta(\omega_0 - m)}{\omega_0^2 - 4\cos^2\theta}.$$

We then combine equations (2.22) and (2.23) into a single differential equation for $\boldsymbol{b}$

$$\left(\frac{\partial^2}{\partial\xi^2} - \mathrm{i}\omega^{\bullet}\right)^2 \boldsymbol{b} + A^2\boldsymbol{b} = 0. \tag{2.24}$$

Since both magnetic fields, $\boldsymbol{b}$ and $\hat{\boldsymbol{B}}$, decay away from the interface, the boundary conditions

$$\boldsymbol{b}(\xi = \infty) = 0, \quad \hat{\boldsymbol{B}}(\hat{\xi} = \infty) = 0 \tag{2.25}$$

can be imposed. It is then straightforward to show that the solution of the boundary layer equation (2.24) satisfying boundary conditions (2.25) is

$$\boldsymbol{b} = \left[\boldsymbol{C}_1\exp\left\{\frac{-(1+S^+\mathrm{i})}{\sqrt{2}}\,|\omega^{\bullet} + A|^{1/2}\xi\right\}\right.$$
$$\left. + \boldsymbol{C}_2\exp\left\{\frac{-(1+S^-\mathrm{i})}{\sqrt{2}}\,|\omega^{\bullet} - A|^{1/2}\xi\right\}\right] \mathrm{e}^{im\phi+\mathrm{i}\omega_0 t} \tag{2.26}$$

and that the exterior boundary solution is given by

$$\hat{\boldsymbol{B}} = \boldsymbol{C}_3\exp\left\{\frac{-(1+S^{\omega}\mathrm{i})}{\sqrt{2}}\,|\omega_0|^{1/2}\hat{\xi}\right\} \mathrm{e}^{im\phi+\mathrm{i}\omega_0 t}, \tag{2.27}$$

where

$$S^+ = \frac{\omega^{\bullet} + A}{|\omega^{\bullet} + A|}, \quad S^- = \frac{\omega^{\bullet} - A}{|\omega^{\bullet} - A|}, \quad S^{\omega} = \frac{\omega_0}{|\omega_0|}.$$

The complex vectors, $\{\boldsymbol{C}_j, j = 1,2,3\}$, which are a function of θ, m and I, have to be determined by the matching conditions at the interface between the fluid and container. The first matching condition is (2.6*a*), which may be rewritten as

$$\hat{\boldsymbol{B}} = \boldsymbol{b} + \boldsymbol{b}_0 \tag{2.28}$$

at the fluid–solid interface. We can express the second boundary condition (2.6*b*), after using Ohm's law, by

$$E_{\lambda}\hat{\boldsymbol{r}} \times \nabla \times (\boldsymbol{b}_0 + \boldsymbol{b}) - I\hat{\boldsymbol{r}} \times [\boldsymbol{u}_0 \times (\hat{\boldsymbol{k}} \times \boldsymbol{r})] = \hat{E}_{\lambda}\hat{\boldsymbol{r}} \times \nabla \times \hat{\boldsymbol{B}}. \tag{2.29}$$

Noting that

$$\hat{\boldsymbol{r}} \times [\boldsymbol{u}_0 \times (\hat{\boldsymbol{k}} \times \boldsymbol{r})] = \hat{\boldsymbol{r}} \times [\hat{\boldsymbol{k}}(\boldsymbol{r}\cdot\boldsymbol{u}_0) - \boldsymbol{r}(\boldsymbol{u}_0\cdot\hat{\boldsymbol{k}})] = 0$$

at $r = 1$ and making use of the properties of a boundary-layer solution, we can simplify the second condition to

$$E_{\lambda}\hat{\boldsymbol{r}} \times \frac{\partial \boldsymbol{b}}{\partial r} = \hat{E}_{\lambda}\hat{\boldsymbol{r}} \times \frac{\partial \hat{\boldsymbol{B}}}{\partial r}. \tag{2.30}$$

It is worth mentioning that the normal components of $\{\boldsymbol{C}_j, j = 1,2,3\}$ cannot be determined by these matching conditions at the interface, although it can be readily shown from conditions (2.28) and (2.30) that

$$\hat{\boldsymbol{r}}\cdot\boldsymbol{C}_1 = \hat{\boldsymbol{r}}\cdot\boldsymbol{C}_2, \quad \hat{\boldsymbol{r}}\cdot\boldsymbol{C}_3 = 2\hat{\boldsymbol{r}}\cdot\boldsymbol{C}_2.$$

The normal components can be expressed in terms of the tangential components through the equation $\nabla\cdot\boldsymbol{b} = 0$ in analogy with the problem of Ekman boundary suction (Greenspan 1969)

$$\hat{\boldsymbol{r}}\cdot\boldsymbol{C}_1 = \frac{-\sqrt{2}E_{\hat{\lambda}}^{1/2}\sin^{-1}\theta}{(1+S^{+}\mathrm{i})|\omega^{*}+A|^{1/2}+(1+S^{-}\mathrm{i})|\omega^{*}-A|^{1/2}}\left[\frac{\partial}{\partial\theta}\sin\theta\hat{\phi}-\mathrm{i}m\hat{\theta}\right]\cdot[\hat{\boldsymbol{r}}\times(\boldsymbol{C}_1+\boldsymbol{C}_2)]. \tag{2.31}$$

It is evident that the contribution of ohmic dissipation in connection with the normal components is of second order compared to the tangential components. Accordingly, the contribution from the normal components $\{\hat{\boldsymbol{r}}\cdot\boldsymbol{C}_j, j=1,2,3\}$ will be neglected. We may also notice that the boundary solution (2.26) breaks down at the critical latitude $\theta_c = \cos^{-1}(\omega_0/2)$. But it is not expected that this singular character of the boundary-layer solution would produce a significant effect on the stability problem, because both the amplitude and the gradient of $\boldsymbol{u}_0$ are weak at θ_c. In the similar problem of convection with the Ekman boundary layer resulting from the non-slip boundary conditions (Zhang 1995*a*), the corresponding analytical solutions have the same singular character but a good quantitative agreement between the analytical solutions and fully numerical solutions is achieved. In the following section, we will use $\{\boldsymbol{C}_j, j=1,2,3\}$, obtained through the matching conditions (2.28), (2.30), to examine the stability characteristics according to equation (2.18).

3. Instabilities driven by the Hartmann boundary layer

3.1. *Perfectly conducting container:* $\hat{\lambda}=0$

The simplest case of the instability problem corresponds to a perfect conducting container, $\hat{E}_{\hat{\lambda}}=0$. This case was studied by Roberts & Loper (1979) and the instability criterion (1.3) was derived. In order to make a comparison with the previous results and also to examine the viscous influence on the instability, the analysis of Roberts & Loper (1979) is extended through the inclusion of the viscous effects associated with the stress-free velocity boundary conditions. Magnetic boundary conditions (2.28) and (2.29) with $\hat{E}_{\hat{\lambda}}=0$ become

$$\hat{\boldsymbol{r}}\cdot(\boldsymbol{b}+\boldsymbol{b}_0)=0,\quad \hat{\boldsymbol{r}}\times\nabla\times(\boldsymbol{b}_0+\boldsymbol{b})=0, \tag{3.1}$$

and the stress-free conditions at $r=1$ may be written as

$$\frac{\partial(u_\phi/r)}{\partial r}=\frac{\partial(u_\theta/r)}{\partial r}=u_r=0. \tag{3.2}$$

Using boundary conditions (3.1) and (3.2), we can express the integrals, $\mathscr{I}_j, j=1,2,3$ in the form

$$\mathscr{I}_1 = \frac{-Im}{\omega_0}\int_V|\nabla\times\boldsymbol{u}_0|^2\mathrm{d}V,$$
$$\mathscr{I}_2 = \frac{-Im(\omega_0^2-I^2m^2)}{2\omega_0(\omega_0-I^2m)}\int_V|\nabla\times\boldsymbol{u}_0|^2\mathrm{d}V,$$
$$\mathscr{I}_3 = -\int_V|\nabla\times\boldsymbol{u}_0|^2\mathrm{d}V.$$

Substitution of $\mathscr{I}_j, j=1,2,3$, into equation (2.18) yields

$$\sigma^{*}=\frac{I^2m(\omega_0-m)-\epsilon\omega_0(\omega_0-I^2m)}{\omega_0^2-2I^2m\omega_0+I^2m^2}, \tag{3.3}$$

where

$$\sigma^{*} = \sigma \int_V |\boldsymbol{u}_0|^2 \mathrm{d}V \Big/ \left(E_\lambda \int_V |\nabla \times \boldsymbol{u}_0|^2 \mathrm{d}V\right).$$

The instability criterion (3.3) is consistent with that (equation (1.3)) of Roberts & Loper if $\epsilon = 0$. To determine the stability of field (2.7), expressions for ω_0 are needed while explicit solutions of $\boldsymbol{u}_0$ and $\boldsymbol{b}_0$ are not required. Let us first consider the equatorially anti-symmetric hydromagnetic modes ($l - m = 1$) with ω_0 given by

$$\omega_0 = \frac{1}{m+1}\left[1 + S(1 - I^2/I_1^2)^{1/2}\right], \tag{3.4}$$

where S is $+1$ or -1 and

$$I_1^2 = \frac{1}{m(m+1)(2 - m^2 - m)}. \tag{3.5}$$

Obviously, $\sigma^{*} < 0$ if $m > 1$ and $\omega_0 < 0$. When $m = 1$ we obtain from (3.3) and (3.4)

$$\sigma = -E_\lambda \int_V |\nabla \times \boldsymbol{u}_0|^2 \mathrm{d}V \Big/ \int_V |\boldsymbol{u}_0|^2 \mathrm{d}V, \quad S = -1$$

and

$$\sigma = -E_\nu \int_V |\nabla \times \boldsymbol{u}_0|^2 \mathrm{d}V \Big/ \int_V |\boldsymbol{u}_0|^2 \mathrm{d}V, \quad S = +1.$$

Consider now the equatorially symmetric hydromagnetic mode ($l - m = 2$). For a given value of the wavenumber m and the field strength parameter I, there exist four different solutions of $\boldsymbol{u}_0$ for this class with the corresponding frequencies

$$\omega_0 = \frac{1 + S_1 \Delta^{1/2}}{m+2}\left[1 + S_2(1 - I^2/I_2^2)^{1/2}\right], \tag{3.6}$$

where S_1 and S_2 are $+1$ or -1 and I_2^2 is

$$I_2^2 = \frac{(m^2 + 6m + 6) + 2S_1(2m+3)\Delta^{1/2}}{m(m+2)(2m+3)(-m^2 - 2m + 2 + S_1 2\Delta^{1/2})} \tag{3.7}$$

with

$$\Delta = \frac{m^2 + 4m + 3}{2m + 3}.$$

For $S_1 = 1, S_2 = 1$, hydromagnetic waves are fast and westward propagating ($\omega_0 > 0$); for $S_1 = 1, S_2 = -1$, the waves propagate eastward ($\omega_0 < 0$) except for $m = 1$, but are slow; for $S_1 = -1, S_2 = -1$, the waves are fast and westward propagating; and for $S_1 = -1, S_2 = 1$ the waves propagate eastward, but are slow.

The relationship between I^2 and ω_0 for the four different types of solutions is presented in figure 1. The most interesting class is when the azimuthal wavenumber $m = 1$ and $S_1 = 1$ (figure 1*a*): both the fast and slow waves propagate westward ($\omega_0 > 0$) and become unstable when $I^2 > I_0^2 = 1.1177$ without the diffusive effects. In this paper we shall mainly focus on this class of solutions but in the region $I^2 < 1.1177$ with $\omega_0 = O(1)$ where the system is not unstable without the effects of a hydromagnetic boundary layer. The growth rate σ^{*} in equation (3.3) can be readily evaluated by using expression (3.6), and is shown in figure 2 for $S_1 = 1, S_2 = 1$ and $m = 1$ as a function of I^2 in the region $I^2 < 1.1177$ for different values of ϵ. When the viscous effect is substantial, $\epsilon = 0.5$, the hydromagnetic system is stable in the region $I^2 < 1$ owing largely to the stabilizing viscous influences in the boundary layer.

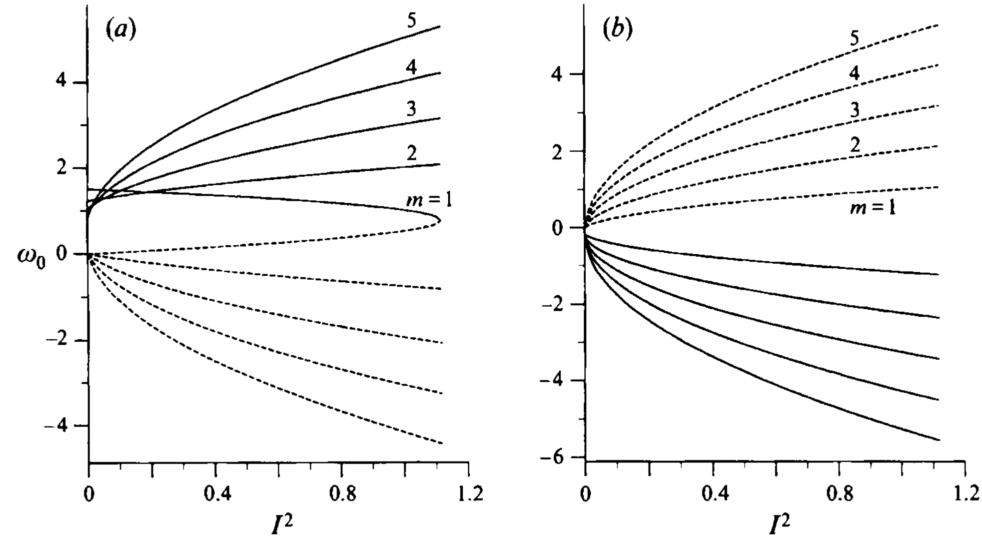

FIGURE 1. The frequencies ω_0 given by equation (3.6) plotted against I^2. Solid lines represent the cases (a) $S_1 = 1, S_2 = 1$ and (b) $S_1 = -1, S_2 = +1$; dashed lines correspond to the cases (a) $S_1 = 1, S_2 = -1$ and (b) $S_1 = -1, S_2 = -1$.

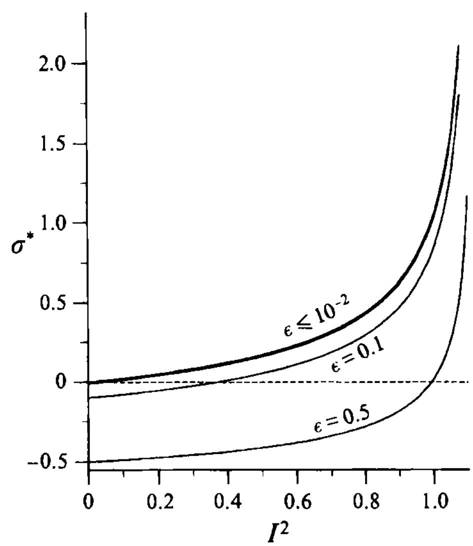

FIGURE 2. The scaled growth rate, σ^*, plotted against I^2 with $m = 1$ and $\hat{\lambda} = 0$ for different values of ϵ.

But the viscosity plays an insignificant role when $\epsilon \leqslant 10^{-2}$ and the hydromagnetic instabilities driven by the boundary-layer effects grow at a rate proportional to E_{λ}. Our result which includes the effect of viscosity and uses explicit solutions for ω_0 in a sphere is consistent with that of Roberts & Loper (1979) for $\epsilon = 0$.

3.2. *Nearly perfectly conducting container: $\hat{\lambda}/\lambda \ll 1$*

Another simple case of the stability problem is obtained when the conductivity of the container is much larger than that of the fluid: $\hat{\lambda}/\lambda \ll 1$. To leading order, the matching conditions (2.28), (2.30) together with the boundary layer solutions (2.26) and (2.27) give

$$\boldsymbol{C}_1 = \frac{-Im}{4\omega_0}\left(\frac{\hat{\lambda}}{\lambda}\right)^{1/2} \frac{|\omega_0|^{1/2}[(1+S^+S^\omega)+\mathrm{i}(S^\omega - S^+)]}{|\omega^* + A|^{1/2}}[\boldsymbol{u}_0 + \mathrm{i}\hat{\boldsymbol{r}} \times \boldsymbol{u}_0]_{r_o}, \tag{3.8}$$

$$\boldsymbol{C}_2 = \frac{-Im}{4\omega_0}\left(\frac{\hat{\lambda}}{\lambda}\right)^{1/2} \frac{|\omega_0|^{1/2}[(1+S^-S^\omega)+\mathrm{i}(S^\omega - S^-)]}{|\omega^* - A|^{1/2}}[\boldsymbol{u}_0 - \mathrm{i}\hat{\boldsymbol{r}} \times \boldsymbol{u}_0]_{r_o}, \tag{3.9}$$

$$\boldsymbol{C}_3 = \boldsymbol{C}_1 + \boldsymbol{C}_2 - \frac{Im}{\omega_0}[\boldsymbol{u}_0]_{r_o}, \tag{3.10}$$

where $[X]_{r_0}$ denotes the evaluation of vector $\boldsymbol{X}$ at $r = 1$. On substituting the boundary-layer solution (2.26) together with (3.8) and (3.9) into equation (2.18) and integrating in the radial direction once, we obtain

$$\sigma = \frac{\hat{E}_\lambda{}^{1/2} I^2 m(\omega_0 - I^2 m)|\omega_0|^{1/2}}{\sqrt{2}\omega_0(\omega_0^2 + I^2m^2 - 2I^2 m\omega_0)} \int_S [-m|\boldsymbol{u}_0|^2 + 2\mathrm{i}\hat{\boldsymbol{k}}\cdot(\boldsymbol{u}_0 \times \boldsymbol{u}_0^*)]\mathrm{d}S \Big/ \int_V |\boldsymbol{u}_0|^2 \mathrm{d}V, \tag{3.11}$$

where analytical expressions in closed form for all integrals, involving both the equatorially symmetric and anti-symmetric modes, can be obtained.

Consider first the equatorially anti-symmetric hydromagnetic modes ($l - m = 1$). Solutions $\boldsymbol{u}_0$ and $\boldsymbol{b}_0$ in this class are purely toroidal, $\hat{\boldsymbol{r}}\cdot\boldsymbol{u}_0 = 0$ and $\hat{\boldsymbol{r}}\cdot\boldsymbol{b}_0 = 0$ (Malkus 1968). Making use of an expression for $\boldsymbol{u}_0$ (for example, equations 20–21 of Zhang 1993), we obtain

$$\int_S |\boldsymbol{u}_0|^2 \mathrm{d}S = \frac{2\pi(2m+2)!!}{m(2m+1)!!}, \quad \int_S 2\mathrm{i}\hat{\boldsymbol{k}}\cdot(\boldsymbol{u}_0 \times \boldsymbol{u}_0^*)\mathrm{d}S = \frac{16\pi(2m-2)!!}{(2m+1)!!}$$

and

$$\int_V |\boldsymbol{u}_0|^2 \mathrm{d}V = \frac{2\pi(2m+2)!!}{m(2m+3)!!}.$$

It follows that the growth rate of disturbances can be explicitly expressed as

$$\sigma = \frac{\hat{E}_\lambda{}^{1/2} I^2 m|\omega_0|^{1/2}(\omega_0 - I^2 m)}{\sqrt{2}\omega_0(\omega_0^2 + I^2m^2 - 2I^2 m\omega_0)} \frac{(2m+3)(2-m^2-m)}{m+1}, \tag{3.12}$$

where ω_0 is given by equation (3.4). For a given value of the field strength I, it is obvious that $\sigma < 0$ if $m > 1$ and that $\sigma = 0$ if $m = 1$. The basic field (2.7) is thus stable with respect to perturbations of equatorially anti-symmetric waves.

Consider now the equatorially symmetric hydromagnetic mode ($l - m = 2$). The corresponding analytical expressions for the integrals in equation (3.11) are too long to be explicitly presented here, although they can be readily obtained from solutions $\boldsymbol{u}_0$ given by equations (13–15) of Zhang (1993). Several examples for the growth rate are shown in figure 3(a) calculated by using equations (3.6) and (3.11). The westward fast propagating ($S_1 = 1, S_2 = 1$) mode with $m = 1$ is always unstable in the limit of vanishing viscosity of the fluid. It is worth noting that the viscous effect is completely neglected in this case, which would shift the region of instability by $O(\epsilon)$, in a similar way as we discussed in the case $\hat{\lambda} = 0$. The hydromagnetic instabilities grow at a rate proportional to $\hat{E}_\lambda$. But for the same class of the wave with $m = 2$, which is also shown in the figure, the perturbations are damped. It can be concluded that, for a nearly perfectly conducting spherical container $\hat{\lambda}/\lambda \ll 1$, the Malkus magnetic field

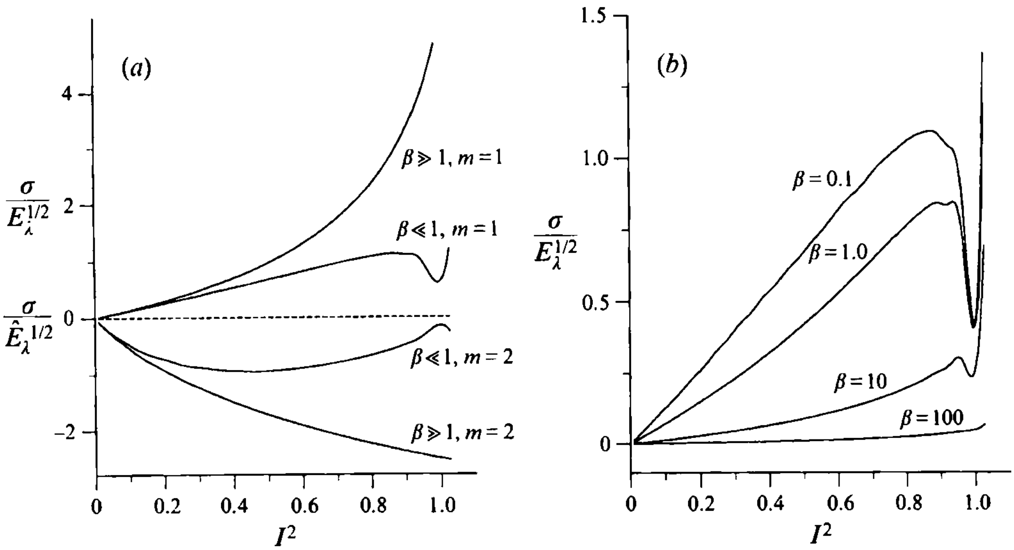

FIGURE 3. (a) The scaled growth rates, $\sigma/\hat{E}_\lambda^{1/2}$ in the case of $\beta = \lambda/\hat{\lambda} \gg 1$ and $\sigma/E_\lambda^{1/2}$ in the case of $\beta \ll 1$, plotted against I^2 with $m = 1$. (b) The scaled growth rates $\sigma/E_\lambda^{1/2}$ plotted against I^2 with $m = 1$ for different values of β.

(2.7) is always unstable in the entire range of the parameter I^2 for which the present analysis is applicable, if the viscous dissipation in the boundary layer is neglected.

3.3. *Nearly insulating container:* $\lambda/\hat{\lambda} \ll 1$

A more complicated, but geophysically relevant case is when the conductivity of the spherical container is much smaller than that of the fluid: $\lambda/\hat{\lambda} \ll 1$. In a first approximation, the matching conditions at $r = 1$ give

$$C_1 = \frac{-Im}{2\omega_0}[\boldsymbol{u}_0 + \mathrm{i}\hat{\boldsymbol{r}} \times \boldsymbol{u}_0]_{r_o}, \quad C_2 = \frac{-Im}{2\omega_0}[\boldsymbol{u}_0 - \mathrm{i}\hat{\boldsymbol{r}} \times \boldsymbol{u}_0]_{r_o}, \quad C_3 = 0. \tag{3.13}$$

Substitution of the corresponding solution of the Hartmann-layer equation into (2.18) yields

$$\sigma = \frac{E_\lambda^{1/2} I^2 m(\omega_0 - I^2 m)}{\sqrt{2}\omega_0(\omega_0^2 + I^2 m^2 - 2I^2 m\omega_0)} \int_S [(-m|\boldsymbol{u}_0|^2 + 2\mathrm{i}\hat{\boldsymbol{k}}\cdot(\boldsymbol{u}_0 \times \boldsymbol{u}_0^*))]|\omega^* + A|^{1/2}\mathrm{d}S \bigg/ \int_V |\boldsymbol{u}_0|^2 \mathrm{d}V. \tag{3.14}$$

In writing this expression, we have neglected the following two integrals:

$$\int_S \mathrm{i}\boldsymbol{u}_0^* \cdot (\hat{\boldsymbol{r}} \times \boldsymbol{u}_0)|\omega^* + A|^{1/2}\mathrm{d}S, \qquad \int_S (\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{r}})|\boldsymbol{u}_0|^2|\omega^* + A|^{1/2}\mathrm{d}S,$$

based on the fact that both the functions $\mathrm{i}\boldsymbol{u}_0^* \cdot (\hat{\boldsymbol{r}} \times \boldsymbol{u}_0)$ and $(\hat{\boldsymbol{k}} \cdot \hat{\boldsymbol{r}})|\boldsymbol{u}_0|^2$ are equatorially anti-symmetric and the function $|\omega^* + A|$ is nearly equatorially symmetric. It has been tested that the inclusion of these two integrals does not produce any noticeable change in the value of the growth rate.

An analytical expression in closed form for equation (3.14) cannot be derived because of the factor $|\omega^* + A|$ in the surface integral. In comparison with the case of a nearly perfectly conducting container, it is apparent that we do not anticipate any fundamental changes in the stability characteristics. The introduction of the positive factor $|\omega^* + A|$ in the surface integral in (3.14) by the nearly insulating boundary condition does not alter the sign of the growth rate. Except for replacing $\hat{E}_\lambda^{1/2}$ by $E_\lambda^{1/2}$, all the main features of the instabilities discussed in §3.2 can therefore be applied to the present case. Instead of $\sigma \sim \hat{E}_\lambda^{1/2}$ in the limit $\hat{\lambda}/\lambda \ll 1$, we have $\sigma \sim E_\lambda^{1/2}$ in the limit $\lambda/\hat{\lambda} \ll 1$. The values of the growth rate as a function of I^2 with $m = 1$ and $m = 2$ for the equatorially symmetric modes are also shown in figure 3(*a*). We can conclude that the Malkus magnetic field (2.7) is always unstable for a nearly insulating spherical container in the limit of vanishing viscosity of the fluid in the entire range of the parameter I^2 for which the present analysis is applicable.

3.4. *Container with arbitrary conductivity*

For the container of arbitrary conductivity, the ohmic dissipation that takes place in the Hartmann layer as well as in the exterior magnetic layer must be taken into account. To leading order, the matching conditions give rise to

$$\frac{C_1}{C^+} = \frac{-Im}{4\omega_0}\frac{|\omega_0|^{1/2}[|\omega^* + A|^{1/2}\beta(1+S^+S^\omega) + 2|\omega_0|^{1/2} + \mathrm{i}|\omega^* + A|\beta(S^\omega - S^+)]}{[|\omega^* + A|\beta^2 + |\omega_0| + |\omega^* + A|^{1/2}\beta|\omega_0|^{1/2}(1+S^+S^\omega)]}, \tag{3.15}$$

$$\frac{C_2}{C^-} = \frac{-Im}{4\omega_0}\frac{|\omega_0|^{1/2}[|\omega^* - A|^{1/2}\beta(1+S^-S^\omega) + 2|\omega_0|^{1/2} + \mathrm{i}|\omega^* - A|\beta(S^\omega - S^-)]}{[|\omega^* - A|\beta^2 + |\omega_0| + |\omega^* - A|^{1/2}\beta|\omega_0|^{1/2}(1+S^-S^\omega)]}, \tag{3.16}$$

where

$$C^+ = [\boldsymbol{u}_0 + \mathrm{i}\hat{\boldsymbol{r}} \times \boldsymbol{u}_0]_{r_o}, \quad C^- = [\boldsymbol{u}_0 - \mathrm{i}\hat{\boldsymbol{r}} \times \boldsymbol{u}_0]_{r_o}, \quad \beta = \lambda/\hat{\lambda}.$$

After inserting the solution for the Hartmann boundary layer into equation (2.18) and integrating in the radial direction once, we obtain

$$\sigma = \frac{E_\lambda^{1/2} I^2 m(\omega_0 - I^2 m)|\omega_0|^{1/2}}{\sqrt{2}\omega_0(\omega_0^2 + I^2m^2 - 2I^2m\omega_0)}$$
$$\times \int_S \left\{ \frac{|\omega^* + A|^{1/2}(|\omega^* + A|^{1/2}\beta + |\omega_0|^{1/2})}{[|\omega^* + A|\beta^2 + |\omega_0| + |\omega^* + A|\beta|\omega_0|^{1/2}(1+S^+S^\omega)]} \right.$$
$$\left. \times\, [(-m - 2\hat{\boldsymbol{k}}\cdot\hat{\boldsymbol{r}})|\boldsymbol{u}_0|^2 + \mathrm{i}(2\hat{\boldsymbol{k}} + m\hat{\boldsymbol{r}})\cdot(\boldsymbol{u}_0 \times \boldsymbol{u}_0^*)] \right\} \mathrm{d}S \bigg/ \int_V |\boldsymbol{u}_0|^2 \mathrm{d}V. \tag{3.17}$$

The surface integrals that have been neglected in the case of nearly insulating containers are now included, though the numerical evaluation with or without them for different values of β still indicates that they make a negligible contribution. An important result emerging from (3.17) is that the different magnetic boundary conditions cannot alter the main features of the instability. This is because the arbitrary conductivity of the container only introduces a factor that is always positive for any values of β. Though the relevant expressions become more complicated, involving the ratio between the diffusivities of the container and of the fluid, the main features of the instability remain unchanged. All conclusions in the previous sections can be applied in principle to the present case after a slight modification. Calculations of the relevant surface integrals have been performed for many different values of β. Apart from the expected variation of time scale of instability ($\sigma \sim \hat{E}_\lambda^{1/2}$ for large

β while $\sigma \sim E_\lambda^{1/2}$ for small β) there are no significant basic changes in the stability properties. For the purpose of comparison, the growth rates calculated by using (3.6) and (3.17) for different values of β are shown in figure 3(*b*) for the symmetric modes. We thus generalize the conclusion given at the end of the previous sections to the case of a spherical container of arbitrary conductivity.

4. Magnetoconvection

In this section we extend the perturbation theory of convection in a rapidly rotating sphere at small Prandtl number (Zhang 1994) by including the effects of the magnetic field (2.7). Without loss of generality, we may introduce the basic temperature field, $T_s(r)$, given by

$$\nabla T_s = -\beta^* \boldsymbol{r}, \tag{4.1}$$

where β^* is a constant related to a uniform distribution of heat source (Chandrasekhar 1961). We shall assume that the spherical container is a perfect conductor and that the velocity boundary conditions are stress-free. The same perturbation expansion as (2.8) in §2 can be used. While the zero-order problem is the same as in §2 in the limit of small Prandtl number (Zhang 1994), the equation of motion in the next order becomes

$$\mathrm{i}\omega_0 \boldsymbol{u}_1 + 2\hat{\boldsymbol{k}} \times \boldsymbol{u}_1 + \nabla P_1 = 2I\hat{\boldsymbol{k}} \times \boldsymbol{b}_1 + \mathrm{i}mI\boldsymbol{b}_1 + Rr\Theta + \boldsymbol{b}_1 - \mathrm{i}\omega_1 \boldsymbol{u}_0 + E_v \nabla^2(\boldsymbol{u}_0 + \boldsymbol{u}_b). \tag{4.2}$$

The other equations (2.13) and (2.14) remain unchanged. A heat equation is thus required to complete the system. The heat equation with the temperature perturbation denoted by Θ and scaled by $\beta^* r_0^2 v/\kappa$, where κ is the thermal diffusivity, may be written as

$$(\mathrm{i}\omega_0 P_r - E_v \nabla^2)\Theta = \boldsymbol{r} \cdot \boldsymbol{u}_0. \tag{4.3}$$

The combination of equations (4.2) and (2.13) and (2.14) gives rise to

$$\mathscr{L}[\boldsymbol{u}_1, \pi_1] = -\mathrm{i}\omega_1 \boldsymbol{u}_0 - \frac{Im\mathrm{i}\omega_1}{\omega_0}\boldsymbol{b}_0 - \frac{2Im\omega_1}{\omega_0}\hat{\boldsymbol{k}} \times \boldsymbol{b}_0 + \boldsymbol{Rr}\Theta$$
$$-\frac{2IE_\lambda \mathrm{i}}{\omega_0}\hat{\boldsymbol{k}} \times \nabla^2(\boldsymbol{b}_0 + \boldsymbol{b}) + \frac{IE_\lambda m}{\omega_0}\nabla^2(\boldsymbol{b}_0 + \boldsymbol{b}) + E_v \nabla^2(\boldsymbol{u}_0 + \boldsymbol{u}_b). \tag{4.4}$$

Imaginary and real parts of the solvability condition yield

$$\omega_1 \int_V \boldsymbol{u}_0^* \cdot \left(\boldsymbol{u}_0 + \frac{Im}{\omega_0}\boldsymbol{b}_0 - \frac{2\mathrm{i}Im}{\omega_0}\hat{\boldsymbol{k}} \times \boldsymbol{b}_0\right) \mathrm{d}V = 0, \tag{4.5}$$

$$E_\lambda \int_V \boldsymbol{u}_0^* \cdot \left[\frac{2\mathrm{i}I}{\omega_0}\hat{\boldsymbol{k}} \times \nabla^2(\boldsymbol{b}_0 + \boldsymbol{b}) - \frac{Im}{\omega_0}\nabla^2(\boldsymbol{b}_0 + \boldsymbol{b}) - \epsilon\nabla^2(\boldsymbol{u}_0 + \boldsymbol{u}_b)\right] \mathrm{d}V = R\int_V \boldsymbol{u}_0^* \cdot \boldsymbol{r}\Theta \mathrm{d}V. \tag{4.6}$$

Upon using equation (2.9), we may write equation (4.5) as

$$\omega_1 \int_V |\boldsymbol{u}_0|^2 \mathrm{d}V \left[\omega_0^2 + I^2 m^2 \omega_0 + \frac{I^2 m^2 (I^2 m^2 - \omega_0^2)}{\omega_0 - I^2 m}\right] = 0. \tag{4.7}$$

Since the expression in the square braket is in general non-zero, the correction for the frequency of convection is zero: $\omega_1 = 0$. The critical Rayleigh number R for the onset of convection is determined by equation (4.6). In the limit of small Prandtl number it can be readily shown that the integral on the right-hand side of equation (4.6) is

$$\int_V \boldsymbol{u}_0^* \cdot \boldsymbol{r}\Theta \mathrm{d}V = E_v \int_V |\nabla\Theta|^2 \mathrm{d}V. \tag{4.8}$$

By a similar analysis to that in §3, equation (4.6) can be reduced to

$$R^* = \epsilon\omega_0^2 + I^2 m^2 - \frac{I^2 m(\omega_0^2 - I^2 m^2)}{\omega_0 - I^2 m}, \tag{4.9}$$

where R^* is the re-scaled Rayleigh number defined as

$$R^* = R\epsilon\omega_0^2 \int_V |\nabla\Theta|^2 \mathrm{d}V \bigg/ \int_V |\nabla \times \boldsymbol{u}_0|^2 \mathrm{d}V. \tag{4.10}$$

With the simple analytical expression (4.9) for the critical Rayleigh number, we are able to identify the precise reason why the negative-Rayleigh-number convection occurs in a rotating fluid sphere in the presence of the field (2.7). In the absence of both the viscous and magnetic dissipations in the boundary layer, $\boldsymbol{u}_b = 0$ and $\boldsymbol{b} = 0$, we obtain the critical Rayleigh number

$$R = 0.$$

In the absence of the magnetic-boundary-layer effect, $\boldsymbol{b} = 0$, equation (4.6) gives rise to the critical Rayleigh number

$$R = E_v \int_V |\nabla \times \boldsymbol{u}_0|^2 \mathrm{d}V \bigg/ \int_V |\nabla\Theta|^2 \mathrm{d}V > 0, \tag{4.11}$$

which is the same as the case of non-magnetic convection (Zhang 1994). In the presence of both the viscous and magnetic dissipations in the boundary layer, $\boldsymbol{u}_b \neq 0$ and $\boldsymbol{b} \neq 0$, the first and second terms on the right-hand side of (4.9) are always positive, providing a sink for the energy by viscous and magnetic dissipation. However, the third term, which is associated with the structure of the magnetic boundary layer strongly modified by rotation, can be negative, releasing energy stored in the magnetic field and thus driving convection solely from the thin spherical boundary layer at the interface between the fluid and solid.

The values of R^* for the onset of convection in equation (4.9) can be readily obtained by using the analytical expression for ω_0 given by equation (3.6). A number of cases for $R^* = R^*(I^2, m = 1)$ are shown in figure 4 for different values of ϵ. For the fast waves, $S_1 = 1$ and $S_2 = 1$, the sign of the Rayleigh number depends mainly on the size of ϵ and I^2: if $\epsilon = 0.1$, $R^* < 0$ for $I^2 < 0.4$ and $R^* > 0$ for $I^2 > 0.4$. But the Rayleigh number is only slightly influenced by the viscosity parameter ϵ for the slow waves, which is evident in equation (4.9).

5. Concluding remarks

We have examined the instability of an electrically conducting fluid in a rapidly rotating fluid sphere with the Malkus magnetic field. It is found that the spherical boundary does not have the stabilizing influence suggested by Roberts & Loper (1979). On the contrary, a new mechanism of hydromagnetic instability driven solely by the spherical Hartmann boundary layer is identified. For $I < I_0$, ohmic dissipation in the spherical boundary layer is the only cause for instabilities of the field configuration (2.7). It follows that caution must be used if one uses the magnetic field (1.1) in a

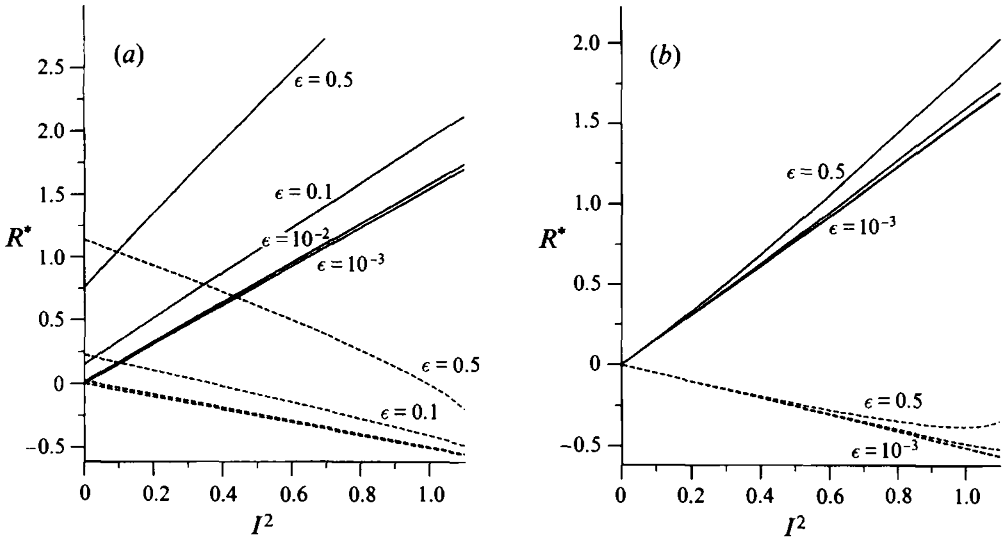

FIGURE 4. The scaled Rayleigh number R^* with $m = 1$ for (a) the equatorially symmetric westward-propagating fast modes and (b) equatorially symmetric eastward-propagating slow modes plotted against the parameter I^2 for different values of ϵ.

spherical geometry and, in particular, if a numerical scheme is involved, because a good resolution of the Hartmann boundary layer will be essential.

An important question is then why Roberts & Loper (1979) did not find the unstable modes in a sphere of arbitrary conductivity. To avoid confusion of notation concerning our Rayleigh number, we denote their parameter R (see equation 3.14, Roberts & Loper 1979) by $\hat{R}$, which is defined as

$$\hat{R} = \frac{2(I^2 m - \omega_0)}{I^2 m^2 - \omega_0^2} \tag{5.1}$$

in our notation. Note also that their index n is the same as our index l and that their ω_0 is the same as our $-\omega_0$. A key for providing an answer to the question is to note that $m = 1, 1 < \hat{R} < 2$, and $\omega_0 > 0$ are not the sufficient conditions for instability in a perfectly conducting spherical container. This can be readily seen by rewriting equation (3.3) in terms of $\hat{R}$:

$$\sigma^* = \frac{I^2(\omega_0 - 1)(2 - \hat{R})}{2\omega_0(1 - \omega_0 - \hat{R}\omega_0 + \hat{R}\omega_0^2)}, \tag{5.2}$$

where $m = 1$ and the viscous term has been neglected by setting $\epsilon = 0$. If we apply the restriction

$$\omega_0 = \tfrac{1}{2}(2 - \hat{R})I^2, \quad I^2 \ll 1, \quad 1 < \hat{R} < 2, \tag{5.3}$$

as Roberts & Loper did in their search for an unstable mode, we obtain the growth rate from equations (5.2) and (5.3)

$$\sigma^* = -1 - \tfrac{1}{2}I^2(2 - \hat{R})\hat{R} + O(I^4), \tag{5.4}$$

where $I^2 \ll 1$ and $1 < \hat{R} < 2$. It follows that the $m = 1$ modes with the restriction (5.3) which are considered by Roberts & Loper are always stable. It is the restriction (5.3) and its subsequent approximation in the analysis of Roberts & Loper (1979) that rules out the instability not only in a spherical container of arbitrary conductivity but also in a perfectly conducting spherical container.

Let us clarify the situation further through the example ($n = 3$ and $m = 1$, the first case in their table 1 and Appendix C) which was studied by Roberts & Loper (1979). For the case $0 < I^2 \ll 1$, $n = 3$ and $m = 1$, there are always four different modes for any given value of I^2. The corresponding frequencies of the modes are given by our equation (3.6). To exclude the two modes that do not give rise to $1 < \hat{R} < 2$ and $\omega_0 > 0$, we set $S_1 = 1$ in equation (3.6). The other two modes with $1 < \hat{R} < 2, \omega_0 > 0$ are then given by $S_2 = -1$ and $S_2 = 1$ with $S_1 = 1$. We first examine the case $S_2 = -1$. Equation (3.6) with $I^2 \ll 1$ and $S_2 = -1$ gives rise to

$$\omega_0 = \frac{I^2}{90}(5 + \sqrt{40})(13 - 2\sqrt{40})(2\sqrt{40} - 5) + O(I^4). \tag{5.5}$$

Substituting ω_0 into (5.1) yields

$$\hat{R} = 2 - \frac{1}{15}(5 + \sqrt{40})(13 - 2\sqrt{40})(2\sqrt{40} - 5) + O(I^2) = \sqrt{40} - 5 + O(I^2), \tag{5.6}$$

which gives rise to $\hat{R} = 1.32455532$ as in their table 1 and in Appendix C. This is the mode studied by Roberts & Loper (1979) which satisfies the restriction (5.3). But this mode is stable for any type of magnetic boundary conditions. We now examine the case $S_2 = +1$. The frequencies ω_0 of the mode can be obtained from equation (3.6) with $I^2 \ll 1$ and $S_2 = 1$:

$$\omega_0 = \tfrac{2}{3}(1 + \sqrt{8/5}) + O(I^2). \tag{5.7}$$

Note that this mode is ruled out by the restriction (5.3) in the analysis of Robert & Loper. Inserting ω_0 into (5.1), we again obtain

$$\hat{R} = \sqrt{40} - 5 + O(I^2). \tag{5.8}$$

Both the modes have exactly the same value of $\hat{R}$ at leading order. However, the mode with equations (5.7) and (5.8) is unstable in the whole range $0 \leqslant \hat{\lambda} < \infty$ and the mode with equations (5.5) and (5.6) is stable in the whole range $0 \leqslant \hat{\lambda} < \infty$. In summary, the restriction (5.3) imposed by Roberts & Loper excludes the unstable modes in a spherical container of arbitrary conductivity, including the perfectly conducting container.

It is also important to note that the time scale used in this paper is much shorter than the magnetic diffusion time scale $\tau_0 = r_0^2/\lambda$, which is normally used (see for example, Fearn 1993 and Zhang & Fearn 1994). When the magnetic diffusion time scale is used, the appropriate non-dimensional parameter is the Elsasser number Λ defined by

$$\Lambda = \frac{{B_0}^2}{\rho\Omega\mu\lambda}. \tag{5.1}$$

The relationship between Λ and I^2 is given by

$$I^2 = \Lambda E_\lambda.$$

The present analysis is based on the limit $E_\lambda \ll 1$ and is valid for $I = O(E_\lambda^{\alpha/4})$, $0 \leqslant \alpha < 1$ (i.e. $\Lambda \gg 1$). As a consequence, our analysis cannot provide the critical value of the Elsasser number Λ required for instability. Fearn (1988) studied the instability problem in a cylinder by solving the complete equations numerically. He found that the critical Elsasser number increases as E_λ decreases. It was shown that the exceptional mode of Robert & Loper in a cylinder is only unstable for Elsasser number much larger than that thought to be present in the Earth's fluid core, leading to the conclusion that the instability found in a cylinder cannot play an important role in the dynamics of the Earth's core (Fearn 1988). But there are no such studies for the instability in a rapidly rotating sphere.

The idea that instabilities can be caused by the effects of a diffusive boundary layer is not new in the context of hydrodynamic stability theory. It is well-known that an inviscid fluid flow can be stable but the corresponding viscous flow may be unstable: viscous diffusion is the cause of the instability. This paper provides an analogy in a spherical hydromagnetic system where the ohmic dissipation in a magnetic boundary layer influenced by rotation is the cause of hydromagnetic instability.

K.Z. is supported by the SERC and a Leverhume grant. We would like to thank the referees for their constructive comments which led to a significant improvement of the paper.

REFERENCES

ACHESON, D. J. 1972 On the hydromagnetic stability of a rotating fluid annulus. *J. Fluid Mech.* **52**, 529–541.

ACHESON, D. J. 1978 Magnetohydrodynamic waves and stabilities in rotating fluids. In *Rotating Fluids in Geophysics* (ed. P. H. Roberts & A. M. Soward), pp. 515–549. Academic.

BUSSE, F. H. 1970 Thermal instabilities in rapidly rotating systems. *J. Fluid Mech.* **44**, 441–460.

BUSSE, F. H. 1982 Thermal convection in rotating systems. *Proc. 9th US Nat. Congress of Appl. Mech.*, pp. 299–305. ASME

CHANDRASEKHAR, S. 1961 *Hydrodynamic and Hydromagnetic Stability.* Clarendon Press.

ELTAYEB, I. A. 1992 The propagation and stability of linear wave motions in rapidly rotating spherical shells: weak magnetic fields. *Geophys. Astrophys. Fluid Dyn.* **67**, 211–240.

ELTAYEB, I. A. & KUMAR, S. 1977 Hydromagnetic convective instability of a rotating, self-gravitating fluid sphere containing a uniform distribution of heat sources. *Proc. R. Soc. Lond.* A **353**, 145–162.

FEARN, D. R. 1979*a* Thermally driven hydrodynamic convection in a rapidly rotating sphere. *Proc. R. Soc. Lond.* A **326**, 227–242.

FEARN, D. R. 1979*b* Thermal and magnetic instabilities in a a rapidly rotating sphere. *Geophys. Astrophys. Fluid Dyn.* **14**, 103–126.

FEARN, D. R. 1983 Hydromagnetic waves in a differentially rotating annulus I. A test of local stability analysis. *Geophys. Astrophys. Fluid Dyn.* **27**, 137–162.

FEARN, D. R. 1988 Hydromagnetic waves in a differentially rotating annulus IV. Insulating boundaries. *Geophys. Astrophys. Fluid Dyn.* **44**, 55–75.

FEARN, D. R. 1993 Magnetic instabilities in rapidly rotating systems. In *Theory of Solar and Planetary Dynamos* (ed. M. R. E. Proctor, P. C. Matthews & A. M. Rucklidge), pp. 59-68. Cambridge University Press.

FEARN, D. R. & PROCTOR, M. R. E. 1983 Hydromagnetic waves in a differentially rotating sphere. *J. Fluid Mech.* **128**, 1–20.

FEARN, D. R. & WEIGLHOFER, W. S. 1991 Magnetic instabilities in rapidly rotating spherical geometries:I. From cylinders to spheres. *Geophys. Astrophys. Fluid Dyn.* **56**, 159–181.

GREENSPAN, H. P. 1969 *The Theory of Rotating Fluids.* Cambridge University Press.

HIDE, R. 1966 Free hydromagnetic oscillations of the Earth's core and the theory of the geomagnetic secular variation. *Phil. Trans R. Soc. Lond.* A **259**, 615–647.

HIDE, R. & STEWARTSON, K. 1972 Hydrodynamics oscillations of the Earth's core. *Rev. Geophys. Space Phys.* **10**, 579–598.

JACOBS, J. A. 1975 *The Earth's Core.* Academic Press.

KERSWELL, R. R. 1994 Tidal excitation of hydromagnetic waves and their damping in the Earth. *J. Fluid Mech.* **274**, 219–241.

LYTTLETON, R. A. 1953 *The Stability of Rotating Liquid Masses.* Cambridge University Press.

MALKUS, W. V. R. 1967 Hydromagnetic planetary waves. *J. Fluid Mech.* **28**, 793–802

MALKUS, W. V. R. 1968 Equatorial planetary waves. *Tellus* **20**, 545–547

PROCTOR, M. R. E. 1994 Magnetoconvection in a rapidly rotating sphere. In *Stellar and Planetary Dynamos* (ed. M. R. E. Proctor & A. D. Gilbert). Cambridge University Press.

ROBERTS, P. H. & LOPER, D. E. 1979 On the diffusive instability of some simple steady magnetohydrodynamic flows. *J. Fluid Mech.* **90**, 641–668.

SOWARD, A. M. 1979 Thermal and magnetically driven convection in a rapidly rotating fluid layer. *J. Fluid Mech.* **90**, 669–684.

ZHANG, K. 1993 On equatorially trapped boundary inertial waves. *J. Fluid Mech.* **248**, 203–217.

ZHANG, K. 1994 On coupling between the Poincaré equation and the heat equation. *J. Fluid Mech.* **268**, 211–229.

ZHANG, K. 1995*a* On coupling between the Poincaré equation and the heat equation:non-slip boundary condition. *J. Fluid Mech.* **284**, 239–256.

ZHANG, K. 1995*b* Spherical shell rotating convection in the presence of a toroidal magnetic field. *Proc R. Soc. Lond.* A **448**, 1–23.

ZHANG, K. & FEARN, D. 1993 How strong is the invisible component of the magnetic field in the Earth's core? *Geophys. Res. Lett.* **20**, 2083–2088.

ZHANG, K. & FEARN, D. R. 1994 Hydromagnetic waves in a rotating spherical shell generated by toroidal magnetic fields. *Geophys. Astrophys. Fluid Dyn.* **77**, 133–157.